ULRICH LINE BUNDLES ON ENRIQUES SURFACES WITH A POLARIZATION OF DEGREE FOUR

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To the memory of Professor Alexandru Lascu

ABSTRACT. In this paper, we prove the existence of an Enriques surface with a polarization of degree four with an Ulrich bundle of rank one. As a consequence, we prove that general polarized Enriques surfaces of degree four, with the same numerical polarization class, carry Ulrich line bundles.

1. INTRODUCTION

Let $X \subset \mathbb{P}^N$ be an n–dimensional smooth projective variety and put $H = \mathcal{O}_X(1)$. An *Ulrich bundle on X* (with respect to the given embedding) [ESW03] is a vector bundle whose twists satisfy a set of vanishing conditions on cohomology:

$$H^i(X, E(-iH)) = 0 \text{ for all } i > 0$$

and

$$H^j(X, E(-(j+1)H)) = 0 \text{ for all } j < n.$$

The presence of twists in the definition shows that this notion strongly depends on the embedding in the projective space. The definition makes sense also for an irreducible variety X, not necessarily smooth [ESW03].

Ulrich bundles were introduced in commutative algebra in relation to maximally-generated maximal Cohen-Macaulay modules [Ul84]. They made a spectacular appearance in algebraic geometry in recent works of Beauville and Eisenbud-Schreyer and their importance is motivated by the relations with the Cayley-Chow forms [Bea00], [ESW03] and with the cohomology tables [ES11].

Let us briefly recall the theory of cohomology tables and how Ulrich bundles appear naturally in this context. For any coherent sheaf $\mathcal{F}$ on the variety X, the cohomology table $CT(\mathcal{F})$ of $\mathcal{F}$ is defined as the table

$\cdots$	$\gamma_{n,-n-1}$	$\gamma_{n,-n}$	$\gamma_{n,-n+1}$	$\cdots$
$\cdots$	$\gamma_{n-1,-n}$	$\gamma_{n-1,-n+1}$	$\gamma_{n-1,-n+2}$	$\cdots$
$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
$\cdots$	$\gamma_{0,-1}$	$\gamma_{0,0}$	$\gamma_{0,1}$	$\cdots$

where $\gamma_{i,j} = h^i(X, \mathcal{F}(j))$ [ES11]. The cohomology table $CT(\mathcal{F})$ is an element of the space $\prod_{-\infty}^{\infty} \mathbb{Q}^{n+1}$. Varying the sheaf on X, since $CT(\mathcal{F} \oplus \mathcal{F}') = CT(\mathcal{F}) + CT(\mathcal{F}')$, the rays of these tables define a cone, called *the cone of cohomology tables* and denoted by $\mathcal{C}(X, \mathcal{O}_X(1))$. Eisenbud and Schreyer proposed a study of this cone, and obtained a nice description in the case of projective spaces [ES11]. In the

general case, we observe that a linear projection $\pi : X \to \mathbb{P}^n$ induces an injective map $\pi_* : \mathcal{C}(X, \mathcal{O}_X(1)) \to \mathcal{C}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. If E is a vector bundle on X such that its direct image on $\mathbb{P}^n$ is trivial, then $\pi^*(\cdot) \otimes E$ is an inverse of π_*, and hence π_* becomes an isomorphism. An application of the Beilinson spectral sequence and of the Leray spectral sequence for the finite map π implies that $\pi_* E$ is trivial if and only if E is Ulrich. One concludes that the cone of cohomology table of X is the same with the cone of the n–dimensional projective space if and only if X carries and Ulrich bundle [ES11] and hence the existence problem of Ulrich bundles becomes very important. From the view-point of the cone of cohomology tables, the rank plays no role in the existence problem, in practice, we try to find Ulrich bundles of the smallest rank possible.

If the given polarization H is a multiple of another polarization H' then the existence of H'-Ulrich bundles implies the existence of H-Ulrich bundles of much larger rank, [ESW03, Proposition 5.4]. This phenomenon justifies a straightforward extension of the definition to take into account also polarizations which are not very ample, see Definition 1. From the cone of cohomology tables view-point, this generalization is a perfectly legitimate action. It has however some deficiencies, one of them being the possible lack of a geometric interpretation of the existence of Ulrich bundles for polarizations which are not very ample. We give one example here. In rank-two, Eisenbud and Schreyer proposed the notion of *special* Ulrich bundles, which are Ulrich bundles on a surface X, of determinant $\mathcal{O}_X(K_X + 3H)$ (H is considered very ample). There is a prominent merit of the existence of special Ulrich bundles. Via [ESW03, Corollary 3.4], a special Ulrich bundle provides a very nice presentation of the Cayley-Chow form of X. Indeed, X admits a Pfaffian Bézout form in Plücker coordinates. If the polarization is not very ample, the Cayley-Chow form might not even exist.

In this short note, we investigate Ulrich bundles on Enriques surfaces with a polarization of degree four. Note that since a degree-4 polarization gives a $4 : 1$ map to $\mathbb{P}^2$, it is obviously not very ample, and hence our setup should be interpreted in the extended context of ample (not very ample) polarizations. We prove that there are Enriques surfaces with polarizations of degree four which carry Ulrich line bundles. We denote by $\mathcal{F}_5$ the moduli space of polarized $K3$ surfaces of degree 8. The locus

$$\mathcal{NL}_{7,12} := \{(X, H_X) \in \mathcal{F}_5 \mid \exists\, M \in \mathrm{Pic}(X) \text{ with } H_X \cdot M = 12, M^2 = 12\} \subset \mathcal{F}_5$$

is an irreducible component of the Noether-Lefschetz locus in $\mathcal{F}_5$ and

$$\mathcal{U} := \{(X, H_X) \mid \exists\, H_X\text{-Ulrich line bundle } M\}$$

is an open subset in $\mathcal{NL}_{7,12}$. The locus of polarized $K3$ surfaces which cover Enriques surfaces can be described as

$$\mathcal{K} := \{(X, H_X) \mid \exists\, \theta : X \to X \text{ fixed-point-free involution such that } H_X \simeq \theta^* H_X\}.$$

In the main result, Theorem 13, we show that the intersection $\mathcal{U} \cap \mathcal{K}$ is non-empty. The proof is completed with the help of the Macaulay2 computer-algebra system. Moreover, the line bundle that we construct is a pullback of a line bundle from the Enriques surface, which turns out to be Ulrich, too.

The outline of the paper is the following. In section 2 we recall a few facts related to Ulrich bundles and on the geometry of Enriques surfaces. In section 3 we prove the existence of an Enriques surface with a polarization of degree four

with an Ulrich bundle of rank one. As already mentioned, the construction uses the existence of an Ulrich line bundle on its $K3$ cover. As a consequence, we prove that a general polarized Enriques surface of degree four with the same numerical polarization carries an Ulrich line bundle, Corollary 14.

2. Preliminaries

2.1. **Ulrich bundles.** In this section we briefly review the definitions and properties of Ulrich bundles. We extend slightly the setup, to allow polarizations which are not very ample:

Definition 1 (compare to [ESW03], Proposition 2.1). Let X be a projective variety of dimension n and H be an ample and globally generated line bundle on X. We say that a vector bundle E on X is H-*Ulrich* (or *Ulrich with respect to H*) if

$$H^i(X, E(-iH)) = 0 \text{ for all } i > 0$$

and

$$H^j(X, E(-(j+1)H) = 0 \text{ for all } j < n.$$

Remark 2. With this definition, if Y is the image of X in $\mathbb{P}V^*$ via the morphism φ given by a linear system corresponding to a space $V \subset H^0(X, H)$ which generates H, a bundle E is Ulrich with respect to H if and only if $\varphi_* E$ is Ulrich on $Y \subset \mathbb{P}V^*$.

In [ESW03], the original definition assumes that the polarization is very ample. The potential of this extended definition is underlined by the following result, see [ESW03, Proposition 5.4 and Corollary 5.7]

Proposition 3. Let $\varphi : X \to \mathbb{P}^n$ be a finite morphism and denote $H_X = \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$. If X carries an Ulrich bundle E with respect to H_X, then X carries an Ulrich bundle with respect to dH_X for any integer $d > 0$.

The existence of Ulrich bundles with respect to multiples of H_X is hence ensured by the existence of H_X-Ulrich bundles, however, the ranks might differ drastically.

Most of the cases known to carry Ulrich bundles in the classical definition continue to have Ulrich bundles also in this extended framework. We discuss below, in Example 4, the curve case which is identical with [ESW03].

Example 4. If X is a curve of genus g, H is an ample and globally generated line bundle on C, and L is an arbitrary line bundle, then L is H-Ulrich if and only if $\deg(L - H) = g - 1$ and $h^0(L - H) = 0$. Hence a general line bundle of degree $\deg(H) + g - 1$ will be H-Ulrich.

In the sequel, we work on a projective surface S.

Definition 5 ([ESW03]). Let S be a projective surface and H be an ample and globally generated line bundle on S. A vector bundle E of rank 2 is called *special Ulrich* if it is 0-regular with respect to H and $\det E = \mathcal{O}_S(K_S + 3H)$.

One can check immediately that a special Ulrich bundle is Ulrich. Also note that any H-Ulrich bundle E on S satisfies

$$H \cdot \left(c_1(E) - \frac{\text{rank}(E)}{2}(K_S + 3H) \right) = 0$$

since $\chi(E(-H)) = \chi(E(-2H)) = 0$, [AFO12]. Hence, special Ulrich bundles are the simplest vector bundles of rank 2 which satisfy the above identity. Eisenbud and Schreyer proved the following structure result:

Proposition 6. [ESW03, Proposition 6.2] *Let $C \in |K_S + 3H|$ be a smooth curve on S and let A be a line bundle on C with*

$$\deg A = \frac{5}{2}H^2 + \frac{3}{2}(K_S \cdot H) + 2\chi(\mathcal{O}_S).$$

If $\sigma_0, \sigma_1 \in H^0(A)$ define a base point free pencil and $H^1(C, A(K_S + H)) = 0$, then the bundle E defined by the Lazarsfeld-Mukai sequence

$$0 \to E^\vee \to \mathcal{O}_S^{\oplus 2} \overset{(\sigma_0, \sigma_1)}{\longrightarrow} A \to 0$$

is a special Ulrich bundle. Conversely, every special rank 2 Ulrich bundle on S can be obtained from a Lazarsfeld-Mukai sequence.

The bundles E from the proposition are called *Lazarsfeld-Mukai* bundles. They have been defined first on $K3$ surfaces [La86, Mu89] and they are connected with several distinct problems involving curves on $K3$ surfaces: Brill-Noether theory, classification of Fano varieties, syzygies etc. They are very natural and interesting objects with applications in several problems. In our specific situation, we see that the Lazarsfeld-Mukai bundles with given Chern classes are the only candidates to be special Ulrich, in view of Proposition 6.

Example 7. Assume S is a $K3$ surface and H_S be a very ample line bundle on S. In [AFO12, Theorem 0.4], the existence of special Ulrich bundles on $K3$ surfaces satisfying a mild Brill-Noether condition is proved. Specifically, it is required that the Clifford index of a general cubic section be computed by H_S. A $K3$ surface whose Picard group is generated by H_S automatically satisfies this hypothesis. In [AFO12], H_S was supposed to be very ample. However, the exactly same proof goes through even if we only assume that H_S is ample and globally generated.

As noted in [AFO12], the sufficient Brill-Noether condition on $K3$ surfaces is used only to ensure the existence of a base-point-free pencil of degree $\frac{5}{2}H_S^2 + 4$ on the cubic sections. However, there are cases not covered by this Brill-Noether condition and which still carry Ulrich bundles, and even special Ulrich bundles.

Example 8. If S is an Enriques surface, and H_S is ample and globally generated, the existence of special Ulrich bundles on S was proved in [Bea16]. In loc.cit. it was assumed that H_S is very ample, however, the proof goes through even under weaker assumptions. In some cases, S carries Ulrich line bundles [BN16]. Borisov and Nuer conjectured that this should always be the case, for very ample polarizations on unnodal Enriques surfaces.

3. Construction of Ulrich bundles using $K3$ covers

In this section we describe how we obtain an Ulrich bundle on an Enriques surface from its $K3$ cover. Let Y be an Enriques surface and H_Y be an ample and globally generated line bundle on Y. It admits an étale $K3$ cover, namely $\sigma : X \to Y$ such that X is a $K3$ surface with a fixed-point-free involution $\theta : X \to X$ which induces $\sigma : X \to X/\theta \simeq Y$.

Recall from Remark 2 that if there is an Ulrich bundle E on X with respect to $H_X := \sigma^* H_Y$, then its push-forward $F = \sigma_* E$ is an Ulrich bundle of rank $2 \cdot rk(E)$ on (Y, H_Y). The main goal of this section is to construct an Ulrich line bundle on a particular (Y, H_Y) occurring as a direct summand of the push-forward of an Ulrich line bundle M on X by σ.

It is natural to ask which polarized $K3$ surfaces (X, H_X) carry an H_X-Ulrich line bundle M equipped with a fixed-point-free involution θ. We divide into smaller questions as follows:

Question 9.

(1) Which $K3$ surface X can cover an Enriques surface Y?
(2) Given such a covering $\sigma : X \to Y$, which H_X can be described as the pull-back of an ample and globally generated line bundle H_Y on Y?
(3) Which polarized $K3$ surfaces (X, H_X) carry Ulrich line bundles?

There is a very nice criterion in [Keu90] which answers the first question. It gives a whole class of $K3$ covers. In this paper, we only use a weaker result:

Theorem 10. [Keu90, Theorem 2] *Every algebraic Kummer surface is the $K3$-cover of some Enriques surface.*

Horikawa's theorem answers the second question. We follow the notation in [Keu90]. Let $\Lambda = U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)$ be the $K3$ lattice. We choose a basis of Λ of the form $v_1, v_2, v_1', v_2', v_1'', v_2'', e_1', \cdots, e_8', e_1'', \cdots, e_8''$ where the first 3 pairs are the standard bases of U and the remaining 2 octuples are the standard bases of $E_8(-1)$. There is an involution $\vartheta : \Lambda \to \Lambda$ given by

$$\vartheta(v_i) = -v_i, \vartheta(v_i') = v_i'', \vartheta(v_i'') = v_i', \vartheta(e_i') = e_i'', \vartheta(e_i'') = e_i'.$$

We denote the ϑ-invariant sublattice by $\Lambda^+ \simeq U(2) \oplus E_8(-2)$. Recall that the covering map $\sigma : X \to Y$ is determined by the choice of a fixed-point-free involution $\theta : X \to X$.

Theorem 11. [Hor78, I, Theorem 5.4] *There is an isometry $\phi : H^2(X, \mathbb{Z}) \to \Lambda$ such that the following diagram*

$$\begin{array}{ccc} H^2(X, \mathbb{Z}) & \xrightarrow{\theta^*} & H^2(X, \mathbb{Z}) \\ \phi \downarrow & & \downarrow \phi \\ \Lambda & \xrightarrow{\vartheta} & \Lambda \end{array}$$

commutes. In particular, ϕ induces an isomorphism

$$\bar{\phi} : H^2(X, \mathbb{Z})^{\theta^*} = \sigma^* H^2(Y, \mathbb{Z}) = \sigma^* \operatorname{Pic}(Y) \to \Lambda^+.$$

Hence, Horikawa's theorem implies that a divisor (equivalently, a line bundle) which is invariant under θ can be obtained by the pull-back of a divisor on Y, and vice versa.

For the last question, there are some numerical conditions which filter out most of line bundles. Let X be a $K3$ surface and H_X be an ample and globally generated line bundle with $H_X^2 = 2s > 0$. If there is an Ulrich line bundle M with respect to H_X, it must satisfy $\chi(M - H_X) = \chi(M - 2H_X) = 0$. Using the Riemann-Roch formula we have

$$H_X \cdot \left(\frac{3}{2} H_X - M \right) = 3s - (H_X \cdot M) = 0,$$

so $(H_X \cdot M) = 3s$. Since M is Ulrich, applying Riemann-Roch formula once again yields

$$2 + \frac{M^2}{2} = \chi(M) = h^0(M) = \deg(X) \cdot \operatorname{rank}(M) = 2s,$$

so $M^2 = 4s - 4$. Note that not all line bundles M with $(M \cdot H_X) = 3s$, $M^2 = 4s - 4$ are Ulrich, see Remark 15.

Lemma 12. *Let X be a K3 surface, and H_X be an ample and globally generated line bundle with $H_X^2 = 2s > 0$. Let M be a line bundle on X with $H_X \cdot M = 3s$ and $M^2 = 4s - 4$. M is an Ulrich line bundle with respect to H_X if and only if both corresponding divisors $M - H_X$ and $2H_X - M$ are not effective.*

Proof. It is enough to show that the 4 cohomology groups

$$H^0(X, M(-H_X)), H^1(X, M(-H_X)), H^1(X, M(-2H_X)), H^2(X, M(-2H_X))$$

vanish simultaneously. By Riemann-Roch formula, $\chi(M - H_X) = 2 + \frac{1}{2}(M - H_X)^2 = 0$ and similarly $\chi(M - 2H_X) = 0$. Since H_X is ample and $H_X \cdot (H_X - M) = -s < 0$, $H_X - M$ cannot be effective, that is, $h^0(X, H_X - M) = h^2(X, M - H_X) = 0$. Similarly, we see that $M - 2H_X$ is not effective, so $h^0(X, M - 2H_X) = 0$. So we have 2 equalities

$$\begin{aligned} h^0(X, M - H_X) &= h^1(X, M - H_X) \\ h^1(X, M - 2H_X) &= h^2(X, M - 2H_X). \end{aligned}$$

Since $h^2(X, M - 2H_X) = h^0(X, 2H_X - M)$, we get the desired result. $\qquad\square$

Before constructing an H_X-Ulrich line bundle on some $K3$ surface X which covers an Enriques surface, we briefly explain why this problem is quite difficult. We denote by $\mathcal{F}_{s+1}$ the moduli space of polarized $K3$ surfaces $\mathcal{F}_{s+1}$ of degree $2s$. The *Noether-Lefschetz locus*, defined as

$$\mathcal{NL} := \{(X, H_X) \mid \mathrm{rk}(\mathrm{Pic}(X)) \geq 2\}$$

is a countable union of divisors inside $\mathcal{F}_{s+1}$. When we fix the number $H_X^2 = 2s$, the locus

$$\mathcal{NL}_{2s-1,3s} := \{(X, H_X) \mid \exists \, M \in \mathrm{Pic}(X) \text{ with } H_X \cdot M = 3s, M^2 = 4s - 4\} \subset \mathcal{F}_{s+1}$$

is an irreducible component of $\mathcal{NL}$ (the subscript $2s - 1$ stands for $\frac{1}{2}M^2 + 1$). Lemma 12 and the semicontinuity of the Ulrich condition in flat families imply that the locus

$$\mathcal{U} := \{(X, H_X) \mid \exists \, H_X\text{-Ulrich line bundle } M\}$$

is an open subset in $\mathcal{NL}_{2s-1,3s}$. Note also that the locus of polarized $K3$ surfaces which cover Enriques surfaces can be described as

$$\mathcal{K} := \{(X, H_X) \mid \exists \, \theta : X \to X \text{ fixed-point-free involution such that } H_X \simeq \theta^* H_X\}$$

which is a closed subset of large codimension (since the Picard number of X is at least 10) in the moduli space of polarized $K3$ surfaces. Hence the problem reduces to finding one element which lies both in an open subset of a Noether-Lefschetz divisor and in a subvariety of large codimension of $\mathcal{F}_{s+1}$.

However, for $s = 4$, we are able to prove that the intersection is nonempty by constructing an explicit example of a $K3$ cover X.

Theorem 13. *When $s = 4$, the intersection $\mathcal{U} \cap \mathcal{K} \subset \mathcal{F}_5$ is nonempty, that is, there is a polarized K3 surface (X, H_X) with $H_X^2 = 8$ which is a K3-cover of an Enriques surface $\sigma : X \to Y$ and carries an $(H_X = \sigma^* H_Y)$-Ulrich line bundle M for some ample line bundle H_Y on Y. Moreover, M can be chosen to be the pull-back of an H_Y-Ulrich line bundle on Y.*

Proof. We proceed in two steps. In the first step, we place ourselves in a more general setup that permits the construction of a class of examples. In the second step we find an explicit example, using Macaulay2.

Step 1. Our candidate $K3$ covers are Kummer surfaces X associated to Jacobian abelian surfaces together with suitable polarizations and potential Ulrich line bundles. Let C be a general curve of genus 2, write $C \to \mathbb{P}^1$ as a double cover of the projective line and denote by $p_1, \ldots, p_6$ the Weierstrass points. They define sixteen theta–characteristics:

$$[p_i], \ i = 1, \ldots, 6, \text{ the odd theta–characteristics, and}$$

$[p_i + p_j - p_k], \ i, j, k = 1, \ldots, 6$ mutually distinct, the even theta–characteristics.

The Jacobian $\mathcal{A} = J(C)$ is an Abelian surface with Néron-Severi group $NS(\mathcal{A}) = \mathbb{Z} \cdot [\Theta]$ with $\Theta^2 = 2$.

The complete linear system $|2\Theta|$ defines a morphism to $\mathbb{P}^3$ and it factors through the singular surface $\mathcal{A}/\iota$, where ι is the involution on $\mathcal{A}$ with 16 fixed points. This embeds $\mathcal{A}/\iota$ as a quartic hypersurface in $\mathbb{P}^3$ with 16 nodes. The Kummer surface $X = Km(\mathcal{A})$ associated to $\mathcal{A}$ is the minimal desingularization of $\mathcal{A}/\iota$. Let $L \in \mathrm{Pic}(X)$ be a line bundle induced by the hyperplane section of the quartic surface $\mathcal{A}/\iota$, and let $E_1, \ldots, E_{16}$ be the 16 exceptional (-2)-curves on X arising from the nodes of $\mathcal{A}/\iota$. By abusing the notation, the curves E_i are usually called in literature *nodes*, too. We have $L^2 = 4, L \cdot E_i = 0$ and $E_i \cdot E_j = -2\delta_{ij}$.

Beside the set of the nodes mentioned above, there is another set of sixteen (-2)–curves, called *tropes* constructed from theta–characteristics, see, for example [Oha09, pag. 175]. Together with the nodes, they form a $(16)_6$ configuration.

We take an ample line bundle $H_X = 2L - \frac{1}{2} \sum_{i=1}^{16} E_i$. Note that H_X induces a smooth projective model of X as the complete intersection of 3 quadrics in $\mathbb{P}^5$ [Shi77, Theorem 2.5], [GS13, Section 5.1]. By choosing suitable coordinates, we may write C as

$$y^2 = \prod_{j=0}^{5}(x - s_j)$$

for some 6-tuple of pairwise distinct numbers $s_j \in \mathbb{C}$. Then its projective model $\varphi_H : X \hookrightarrow \mathbb{P}^5$ is defined by the equations

$$\begin{cases} z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = 0 \\ s_0 z_0^2 + s_1 z_1^2 + s_2 z_2^2 + s_3 z_3^2 + s_4 z_4^2 + s_5 z_5^2 = 0 \\ s_0^2 z_0^2 + s_1^2 z_1^2 + s_2^2 z_2^2 + s_3^2 z_3^2 + s_4^2 z_4^2 + s_5^2 z_5^2 = 0 \end{cases}$$

in $\mathbb{P}^5$ [Shi77, Theorem 2.5]. Note that there are ten fixed-point-free involutions given by changing the sign of three coordinates, for example

$$\theta : (z_0, z_1, z_2, z_3, z_4, z_5) \mapsto (-z_0, -z_1, -z_2, z_3, z_4, z_5),$$

and these involutions correspond to the ten even theta-characteristics [Mu12, p. 233], [Oha09, p. 166]. Reordering the Weierstrass points if necessary, we may assume that the particular involution θ which changes the signs of the first three coordinates corresponds to the theta–characteristic $\beta = [p_4 + p_5 - p_6]$.

The involution θ defined above induces the covering map over an Enriques surface $\sigma : X \to Y = X/\theta$. We can easily check that H_X is θ-invariant as follows. Note that H_X can be represented as a hyperplane divisor of $X \subset \mathbb{P}^5$. For instance, we take the hyperplane section $Z := \{z_0 = 0\} \cap \varphi_{H_X}(X) \subset \mathbb{P}^5$ and see immediately that Z

is θ-invariant. In other words, $c_1(H_X)$ lies on θ-invariant lattice $\Lambda^+ \subset H^2(X, \mathbb{Z})$. Therefore, Horikawa's theorem implies $H_X = \sigma^* H_Y$ for some line bundle H_Y on Y.

Since $H_X = \sigma^* H_Y$ is ample, we see that H_Y is also ample by Nakai-Moishezon criterion. By Riemann-Roch formula, we have $h^0(Y, H_Y) = h^0(Y, K_Y + H_Y) = 3$, which means that H_Y gives rise to a 4-fold cover of $\mathbb{P}^2$.

Following [Oha09], we relabel the nodes by the corresponding 2–torsion points in $\mathcal{A}$:

$$
\begin{aligned}
E_0 &= & \text{node corresponding to } [0] \in \mathcal{A}; \\
E_{ij} = E_{[p_i - p_j]} &= & \text{node corresponding to } [p_i - p_j] \in \mathcal{A}, 1 \le i < j \le 6.
\end{aligned}
$$

The tropes are labelled using their associated theta–characteristics [Oha09], e.g. $T_i = T_{[p_i]}$ corresponds to $[p_i]$ and $T_{ijk} = T_{[p_i + p_j - p_k]}$ corresponds to $[p_i + p_j - p_k]$ for any $i < j < k$. Obviously, if $\{i, j, k\} \cup \{\ell, m, n\} = \{1, \ldots, 6\}$ then $T_{ijk} = T_{\ell mn}$.

Since the fixed-point-free involution θ is a "switch" induced by the even theta characteristic $\beta = [p_4 + p_5 - p_6]$, it swaps the nodes E_α and the tropes $T_{\alpha+\beta}$ in the following way, [Oha09, Section 4, Section 5]:

Nodes		Tropes	Nodes		Tropes
E_0	$\leftrightarrow$	T_{456}	E_{25}	$\leftrightarrow$	T_{246}
E_{12}	$\leftrightarrow$	T_3	E_{26}	$\leftrightarrow$	T_{136}
E_{13}	$\leftrightarrow$	T_2	E_{34}	$\leftrightarrow$	T_{356}
E_{14}	$\leftrightarrow$	T_{156}	E_{35}	$\leftrightarrow$	T_{346}
E_{15}	$\leftrightarrow$	T_{146}	E_{36}	$\leftrightarrow$	T_{126}
E_{16}	$\leftrightarrow$	T_{236}	E_{45}	$\leftrightarrow$	T_6
E_{23}	$\leftrightarrow$	T_1	E_{46}	$\leftrightarrow$	T_5
E_{24}	$\leftrightarrow$	T_{256}	E_{56}	$\leftrightarrow$	T_4

where the corresponding tropes are computed by (see [Oha09, Lemma 4.1])

$$
T_i = \frac{1}{2}\left(L - E_0 - \sum_{k \ne i} E_{ik}\right)
$$

for $1 \le i \le 6$ and

$$
T_{ij6} = \frac{1}{2}(L - E_{i6} - E_{j6} - E_{ij} - E_{\ell m} - E_{mn} - E_{\ell n})
$$

for $1 \le i < j \le 5$, where $\{l, m, n\}$ is the complement of $\{i, j\}$ in $\{1, 2, 3, 4, 5\}$.

Note that, since

$$
L = 2T_6 + E_0 + E_{16} + E_{26} + E_{36} + E_{46} + E_{56},
$$

we obtain the formula

$$
(1) \qquad \theta^*(L) = 3L - E_0 - \sum E_{ij}.
$$

Put $M = 3L - (E_0 + E_{16} + E_{26} + E_{36} + E_{46} + E_{56} + E_{12} + E_{13} + E_{14} + E_{15} + E_{24} + E_{35})$. A direct computation using (1) shows that $\theta^* M = L + T_6 + T_1 + T_{246} + T_{356} = M$, that is, M is invariant under θ^*. Hence, we conclude that $M = \sigma^* N$ for some line bundle N on Y, and $F = \sigma_*(M) = N \oplus (N \otimes K_Y)$.

Remark that $M \cdot H_X = M^2 = 12$. Hence, in view of Lemma 12, this particular line bundle M is Ulrich if and only if the divisors $M - H_X$ and $2H_X - M$ are not effective.

Step 2. Using Macaulay2, see [GS], we provide an example of a polarized $K3$ cover as above, with $M - H_X$ and $2H_X - M$ non–effective. We take the explicit equation for the Kummer quartic surface in $\mathbb{P}^3$ for a genus 2 curve from [Fly93, Section 2]. We also refer [Kum08, Section 4] for more analysis on nodes and tropes. Let C be the hyperelliptic curve given by the equation $y^2 = (x - 1)(x + 1)(x - 2)(x + 2)(x - 3)(x + 3)$. The corresponding equation which gives a Kummer quartic with 16 nodes is the following, [Kum08, Section 4.2]:

```
Macaulay2, version 1.8.2
with packages: ConwayPolynomials, Elimination, IntegralClosure,
               LLLBases, PrimaryDecomposition, ReesAlgebra,
               TangentCone

i1 : S=ZZ/32003[X,Y,Z,W];
i2 : f=7056*X^4-2016*X^2*Y^2+144*Y^4-288*X*Y^2*Z+2888*X^2*Z^2
-196*Y^2*Z^2+56*Z^4+144*X^3*W-196*X^2*Z*W+56*X*Z^2*W-4*Z^3*W
+Y^2*W^2-4*X*Z*W^2;
i3 : I=ideal f;
o3 : Ideal of S
```

We can easily verify that it is a singular surface with 16 distinct nodes as follows.

```
i4 : NODES=ideal singularLocus Proj (S/I);
o4 : Ideal of S
i5 : codim NODES
o5 = 3
i6 : degree NODES
o6 = 16
```

We are interested in the vanishing $H^0(2H_X - M) = H^0(L - E_{23} - E_{25} - E_{34} - E_{45}) = 0$. To compute the cohomology H^0 passing by the map $\phi_{|L|} : X \to \mathbb{P}^3$, we need to pick 4 nodes in the image corresponding to $E_{23}, E_{25}, E_{34}, E_{45}$. Following the computations in [Kum08, Section 4.2], we have 4 points in $\mathbb{P}^3$

$$
\begin{aligned}
p_{23} &= (1 : 1 : -2 : -44) \\
p_{25} &= (1 : 2 : -3 : -42) \\
p_{34} &= (1 : 0 : -4 : -65) \\
p_{45} &= (1 : 1 : -6 : -84).
\end{aligned}
$$

which are 4 nodes of the Kummer quartic $\bar{X} = V(f)$.

Let $J1$ be the ideal for 4 nodes $\{p_{23}, p_{25}, p_{34}, p_{45}\}$, and $J2$ be the ideal for complementary 12 nodes. We chose the ideal manually among the minimal prime ideals to reduce hand-written computations. For practical reasons, we consider also some intermediate saturation processes.

```
i7  : LIST=minimalPrimes NODES
o7  = {ideal (Z, Y, X), ideal (- 14238Z + W, Y, X + 3556Z),
      ----------------------------------------------------------
      ideal (- 8017Z + W, Y, X + 8001Z), ideal (- 50Z + W, Y, X
      ----------------------------------------------------------
      + Z), ideal (- 14Z + W, Y + 16000Z, X + 16001Z), ideal (-
      ----------------------------------------------------------
      14Z + W, Y - 16000Z, X + 16001Z), ideal (- 14Z + W, Y +
      ----------------------------------------------------------
      10667Z, X + 10668Z), ideal (- 14Z + W, Y - 10667Z, X +
      ----------------------------------------------------------
      10668Z), ideal (- 14Z + W, Y + 5334Z, X + 5334Z), ideal (-
      ----------------------------------------------------------
      14Z + W, Y - 5334Z, X + 5334Z), ideal (- 2Z + W, Y -
      ----------------------------------------------------------
      10669Z, X - 10668Z), ideal (- 2Z + W, Y + 10669Z, X -
      ----------------------------------------------------------
      10668Z), ideal (10Z + W, Y - 5333Z, X - 5334Z), ideal (10Z
      ----------------------------------------------------------
      + W, Y + 5333Z, X - 5334Z), ideal (- 22Z + W, Y + 16001Z,
      ----------------------------------------------------------
      X - 16001Z), ideal (- 22Z + W, Y - 16001Z, X - 16001Z)}
o7  : List
i8  : Ip23=LIST_15
o8  = ideal (- 22Z + W, Y - 16001Z, X - 16001Z)
o8  : Ideal of S
i9  : Ip25=LIST_5
o9  = ideal (- 14Z + W, Y - 16000Z, X + 16001Z)
o9  : Ideal of S
i10 : Ip34=LIST_2
o10 = ideal (- 8017Z + W, Y, X + 8001Z)
o10 : Ideal of S
i11 : Ip45=LIST_8
o11 = ideal (- 14Z + W, Y + 5334Z, X + 5334Z)
o11 : Ideal of S
i12 : J1=saturate(Ip23*Ip25*Ip34*Ip45);
o12 : Ideal of S
i13 : Temp1=saturate(LIST_0*LIST_1*LIST_3*LIST_4*LIST_6*LIST_7);
o13 : Ideal of S
i14 : Temp2=saturate(LIST_9*LIST_10*LIST_11*LIST_12*LIST_13*LIST_14);
o14 : Ideal of S
i15 : J2=saturate(Temp1*Temp2);
o15 : Ideal of S
```

Now, the element in $|2H_X - M| = |L - E_{23} - E_{25} - E_{34} - E_{45}|$ corresponds to a hyperplane section passing through $p_{23}, p_{25}, p_{34}, p_{45}$, and we can check that there is no such a hyperplane section:

```
i16 : HH^0(sheaf(S^{1}**module(J1)))
o16 = 0

        ZZ
o16 : ------module
      32003
```

Similarly, if $|M - H_X|$ is nonempty, then $|2(M - H_X)| = |2L + E_{23} + E_{25} + E_{34} + E_{45} - (E_0 + E_{16} + E_{26} + E_{36} + E_{46} + E_{56} + E_{12} + E_{13} + E_{14} + E_{15} + E_{24} + E_{35})|$ is also nonempty. Note that, since $L|_{E_{ij}} \cong \mathcal{O}_{E_{ij}}$ and $\mathcal{O}_{E_{ij}}(E_{ij}) \cong \mathcal{O}_{E_{ij}}(-2)$, we obtain immediately an isomorphism

$$H^0(2L) \cong H^0(2L + E_{23} + E_{25} + E_{34} + E_{45})$$

and hence we can identify $|2L + E_{23} + E_{25} + E_{34} + E_{45} - (E_0 + E_{16} + E_{26} + E_{36} + E_{46} + E_{56} + E_{12} + E_{13} + E_{14} + E_{15} + E_{24} + E_{35})|$ with $|2L - (E_0 + E_{16} + E_{26} + E_{36} + E_{46} + E_{56} + E_{12} + E_{13} + E_{14} + E_{15} + E_{24} + E_{35})|$. Via the map $\varphi_{|L|}$ to $\mathbb{P}^3$, an element in this linear system corresponds to a quadric hypersurface passing through the 12 complementary nodes to $\{p_{23}, p_{25}, p_{34}, p_{45}\}$. Macaulay2 computation shows however that

```
i17 : HH^0(sheaf(S^{2}**module(J2)))
o17 = 0

        ZZ
o17 : ------module
      32003
```

i.e. there is no such a quadric section.

Conclusion. For the example found in the second step, since M is H_X–Ulrich, it follows that F is H_Y–Ulrich, and hence the direct summand N is an H_Y–Ulrich line bundle as well. □

In what follows, we prove the existence of Ulrich line bundles for general Enriques surfaces with a polarization of degree four. We fix some notation. Let $h \in U \oplus E_8(-1)$ be the numerical class of the polarization H_Y constructed above, $\mathcal{M}^0_{En}$ be the 10–dimensional moduli space of Enriques surfaces [Hor78, Nam85, GH16], and $\mathcal{M}^0_{En,h}$ be the moduli space of Enriques surfaces with a polarization of type h, [GH16]. It is also 10–dimensional and irreducible, and there is a natural forgetful morphism $\varphi : \mathcal{M}^0_{En,h} \to \mathcal{M}^0_{En}$ obtained from the descriptions of the two moduli spaces as (open subsets of) quotients of the same bounded domain, [GH16, pag. 59, 61]. We prove:

Corollary 14. *A general polarized Enriques surface* $(Y, H_Y) \in \mathcal{M}^0_{En,h}$ *carries an* H_Y*-Ulrich line bundle.*

Proof. In Theorem 13 we constructed a polarized Enriques surface (Y, H_Y) of degree four and an H_Y-Ulrich line bundle N on it. Note that N satisfies numerical conditions $N^2 = N \cdot H_Y = 6$ since $\chi(N - H_Y) = \chi(N - 2H_Y) = 0$. Denote by $\eta \in U \oplus E_8(-1)$ its numerical class.

We claim that a general Enriques surface $(Y, H_Y) \in \mathcal{M}_{En,h}^0$ has an Ulrich line bundle. Consider the locus

$$\mathcal{U}_{En,h} := \{(Y, H_Y) \mid \exists \text{ an } H_Y\text{-Ulrich line bundle } N\}$$

inside the moduli space $\mathcal{M}_{En,h}^0$ of polarized Enriques surfaces of degree four. Since Ulrich conditions are open in flat families, it is an open subset of the locus

$$\mathcal{NL}_{En,h} := \{(Y, H_Y) \in \mathcal{M}_{En,h}^0 \mid \exists \text{ a line bundle } N \text{ such that } N^2 = N \cdot H_Y = 6\}.$$

We claim that $\mathcal{NL}_{En,h}$ coincides with the whole space $\mathcal{M}_{En,h}^0$. If it happens, then $\mathcal{U}_{En,h}$ is a nonempty open subset of $\mathcal{M}_{En,h}^0$, hence, a general polarized Enriques surface carries an Ulrich line bundle.

Note that for any Enriques surface Y, there exists a polarization H of numerical class h and a line bundle N such that $H^2 = 4$, $N^2 = N \cdot H = 6$. Indeed, any line bundle N of numerical class η satisfies this condition. Via the surjective morphism $\varphi : \mathcal{M}_{En,h}^0 \to \mathcal{M}_{En}^0$, $\mathcal{NL}_{En,h}$ is dominant over $\mathcal{M}_{En}^0$. Since $\mathcal{M}_{En,h}^0$ is an irreducible variety of dimension 10 and $\mathcal{NL}_{En,h} \subseteq \mathcal{M}_{En,h}^0$ is a closed algebraic subset which dominates $\mathcal{M}_{En}^0$ via the map φ, we conclude that $\dim \mathcal{NL}_{En,h} = 10$ and hence $\mathcal{NL}_{En,h}$ and $\mathcal{M}_{En,h}^0$ coincide. $\qquad\square$

Remark 15. Finding an H_X-Ulrich line bundle M for an arbitrary X is not a simple question. Indeed, there is a line bundle M with $H_X \cdot M = M^2 = 12$ which is not Ulrich. Suppose that 8 exceptional curves $E_{i_1}, \ldots, E_{i_8}$ forms an *even eight*, that is, $\sum_{k=1}^8 E_{i_k}$ is divisible by 2 in $\mathrm{Pic}(X)$. Then $M = 2L - \frac{1}{2}\sum_{k=1}^8 E_{i_k}$ satisfies the numerical conditions in Lemma 12. However, we can check it directly that M cannot be Ulrich. We have $M - H_X = \frac{1}{2}\sum_{k=1}^8 E_{j_k}$, where the index set $\{j_1, \ldots, j_8\}$ is the complementary set of $\{i_1, \ldots, i_8\}$ in $\{1, 2, \ldots, 16\}$. By Nikulin [Nik75, Corollary 5], the set $\{j_1, \ldots, j_8\}$ also induces an even eight, so $M - H_X$ is effective and M is not Ulrich.

Remark 16. Corollary 14 can be accounted an evidence of Borisov-Nuer conjecture, even though the authors formulated it for unnodal Enriques surfaces of degree ≥ 10.

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